JOURNAL OF COMPUTATIONAL PHYSICS 119, 388–389 (1995)

NOTE

On Random Walk Models with Space Varying Diffusivity

1. INTRODUCTION

Random walk models with constant diffusivity are often used to model dispersion processes in the environment. In this case it is easy to show that the results of a random walk model are equivalent with the solution of an advection diffusion equation. Generalising the random walk models to a space varying diffusivity is not trivial. Simply using the standard model with space varying variance of the random step produces erroneous results. Hunter, Craig, and Phillips [3] developed some ideas to improve the random walk models in the case of space varying diffusivity. Their work increases the insight into this problem considerably. However they, among many other researchers that develop random walk models, do not seem to be aware of the theory of stochastic differential equations. Using this theory it is possible to derive, for any random walk model, the corresponding advection diffusion equation and vice versa. Starting, for example, with an advection diffusion equation with space varying diffusivity, the random walk model that is exactly consistent with this advection diffusion equation can be obtained. Any other random walk model with space varying diffusivity is not correct and may produce erroneous results.

2. STOCHASTIC DIFFERENTIAL EQUATIONS

Consider the following stochastic differential equation to describe the irregular movement of a particle injected in the fluid at time t_0, at position $X_{t_0} = x_0$ [4, 1],

$$dX_t = f(X_t, t)\,dt + G(X_t, t)\,dW_t, \tag{1}$$

where the vector $f(X_t, t)$ and the matrix $G(X_t, t)$ are, in general, smooth nonlinear functions and the vector W_t represents Brownian motion. This is a Gaussian stochastic process with independent increments characterized by

$$E\{dW_t\} = 0 \tag{2}$$

$$E\{dW_t\,dW_t^{\mathrm{T}}\} = I\,dt, \tag{3}$$

where I is the identity matrix. In case G depends on X_t, the stochastic differential equation (1) is not uniquely defined yet. However, Îto constructed a mathematical interpretation of (1) [4]. As a result (1) is called an Îto equation.

It can be derived that the probability density function $p(x, t)$ to find a particle at position x at time t is given by the Îto Fokker–Planck equation [4],

$$\frac{\partial p}{\partial t} = -\sum_i \frac{\partial f_i p}{\partial x_i} + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 (GG^{\mathrm{T}})_{ij} p}{\partial x_i \partial x_j} \tag{4}$$

with the initial condition

$$p(x, t_0) = \delta(x_0 - x). \tag{5}$$

3. RANDOM WALK MODEL

Equation (4) is an advection-diffusion type equation. As a result we can interpretate an advection diffusion equation as a Fokker–Planck equation. Hence a stochastic differential equation can be derived that is exactly consistent with this advection diffusion equation. Consider for instance the model analysed by Hunter, Craig, and Phillips [3],

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} K \frac{\partial c}{\partial x}. \tag{6}$$

The corresponding (Îto) random walk model is

$$dX_t = \frac{\partial K}{\partial x}\,dt + G\,dW_t, \tag{7}$$

where G is a matrix that satisfies $2K = GG^{\mathrm{T}}$ [4]. This can easily be verified by substituting (7) into the Fokker–Planck equation (6).

The stochastic differential equation can be simulated by approximating this equation numerically [5]. In this way a discrete random walk model can be obtained:

$$X_{k+1} = X_k + \frac{\partial K}{\partial x}\Delta t + G\Delta W_k. \tag{8}$$

Here we have used as an example the Euler scheme with X_k as the numerical approximation of X_{t_k} and ΔW_k as a random variable with zero mean and variance Δt. This scheme is consistent with the Îto definition of the stochastic differential equation (1). By simulating the random walk model (8) for many parti-

0021-9991/95 $12.00

cles, the particle distribution approximates the concentration $c(x, t)$ and will satisfy asymptotically for $N \to \infty$ and $\Delta t \to 0$ the advection diffusion equation (6). From Eq. (7) we see that the "apparent advection velocity" introduced by Hunter, Craig, and Phillips turned out to be $\partial K/\partial x$.

This result is not restricted to cases in which the diffusivity varies in only one direction. Moreover, the expression is exact. Any other additional drift that is introduced in the random walk model will introduce incorrect results for small values of Δt and large number of particles. Another example of a random walk model that can be used to model dispersion processes in shallow water and that is consistent with the vertical integrated two-dimensional advection diffusion equation can be found in Heemink [2].

The results just described are based on a definition of the stochastic differential equation (1) in the Îto sense. However, interpretating this equation differently would also lead to a correct random walk model. If we would, for example, adopt the Stratonovitz definition [4], the Fokker–Planck equation (4) would be different. As a result interpreting this (Stratonovitz) Fokker–Planck equation as the given advection-diffusion equation (6), the corresponding stochastic differential equation would be different from the Îto equation (7). However, by introducing a numerical scheme, that now has to be consistent with the Stratonovitz definition, a correct random walk model can be obtained again [5].

4. CONCLUDING REMARKS

Many researchers that develop random walk models avoid the implementation of models with space varying diffusivity or implement erroneous models. Hunter, Craig, and Phillips [3] have realised this and have developed some ideas to deal with space varying diffusivity. In our note we suggest using the theory of nonlinear stochastic differential equations to derive random walk models in this case. In this way the exact results can be obtained very elegantly. Furthermore, this approach can be used to derive random walk models in many other cases.

REFERENCES

1. L. Arnold, *Stochastic Differential Equations. Theory and Applications* (Wiley, London, 1974).
2. A. W. Heemink, *Stochastic Hydrol. Hydraul.* **4,** 161 (1990).
3. J. R. Hunter, P. D. Craig, and H. E. Phillips, *J. Comput. Phys.* **106,** 366 (1993).
4. A. Jazwinski, *Stochastic Processes and Filtering Theory* (Academic Press, New York, 1970).
5. P. E. Kloeden and E. Platen, Applications of Mathematics, Vol. 23 (Springer Verlag, New York, 1992).

Received September 27, 1993; revised December 15, 1994

Arnold W. Heemink
Delft University of Technology
Faculty of Technical Mathematics and Informatics
Department of Applied Analysis
P.O. Box 5031, 2600 GA Delft, The Netherlands

Pieter A. Blokland
Rijkswaterstaat, Tidal Waters Division
P.O. Box 20907
2500 EX The Hague, The Netherlands